Book

A Simplified Approach
to

Data Structures

Prof.(Dr.)Vishal Goyal, Professor, Punjabi University Patiala
Dr. Lalit Goyal, Associate Professor, DAV College, Jalandhar
Mr. Pawan Kumar, Assistant Professor, DAV College, Bhatinda

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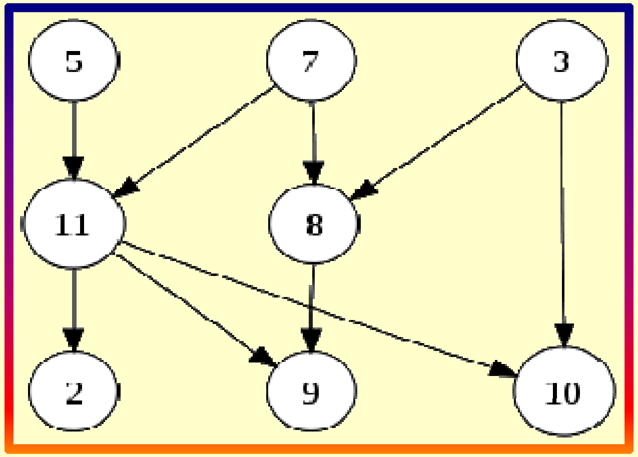
Applications of the Graph

Applications of the Graph

- Finding the reachability
- Finding the shortest path
- Spanning Trees



A labeled simple graph:-
Vertex set $V = \{2,3,5,7,8,9,10,11\}$
Edge set $E = \{\{3,8\}, \{3,10\}, \{5,11\},\{7,8\}, \{7,11\}, \{8,9\}, \{11,2\},\{11,9\},\{11,10\}\}$.

Reachability

•It means that whether a particular vertex is reachable from other vertices of the graph or not.

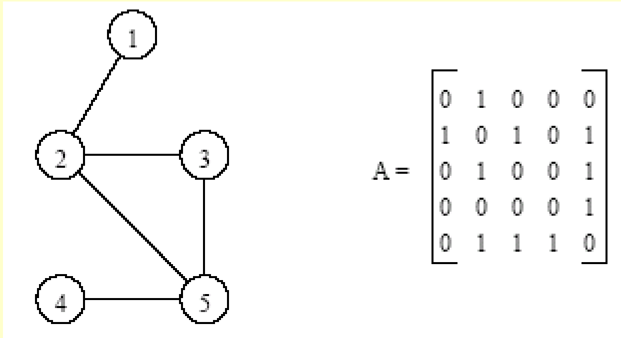
•With the help of reachability matrix of a graph,we can find which vertex of a graph is reachable from which vertex of a graph by 2 ways:-

- ❖ Matrix Multiplication Method
- ❖ Warshall's Algorithm

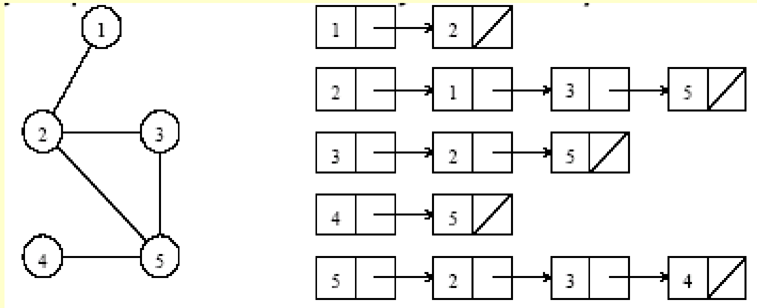
Adjacency Matrix and Adjacency List

Adjacency Matrix:-

The standard adjacency matrix stores a matrix as a 2-D array with each slot in A[i][j] being a 1 if there is an edge from vertex i to vertex j, or storing a 0 otherwise.

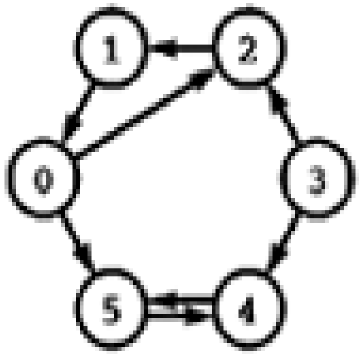


An undirected graph and its adjacency matrix representation.



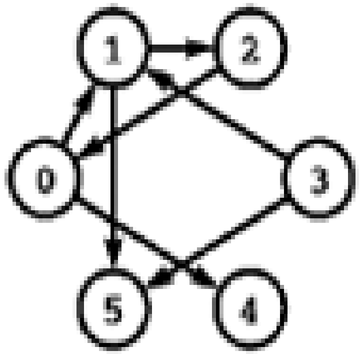
An undirected graph and its adjacency list representation.

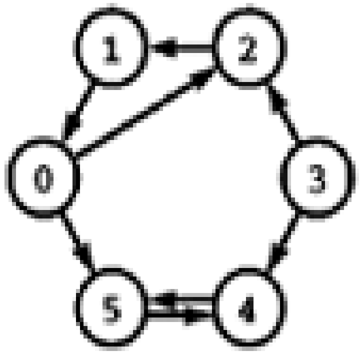
Matrix Multiplication Method



	0	1	2	3	4	5
0	0	0	1	0	0	1
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	0	1	0	1	0
4	0	0	0	0	0	1
5	0	0	0	0	1	0

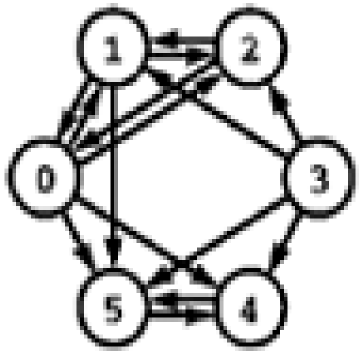
	0	1	2	3	4	5
0	0	1	0	0	1	0
1	0	0	1	0	0	1
2	1	0	0	0	0	0
3	0	1	0	0	0	1
4	0	0	0	0	1	0
5	0	0	0	0	0	1





	0	1	2	3	4	5
0	1	0	1	0	0	1
1	1	1	0	0	0	0
2	0	1	1	0	0	0
3	0	0	1	1	1	0
4	0	0	0	0	1	1
5	0	0	0	0	1	1

	0	1	2	3	4	5
0	1	1	1	0	1	1
1	1	1	1	0	0	1
2	1	1	1	0	0	0
3	0	1	1	1	1	1
4	0	0	0	0	1	1
5	0	0	0	0	1	1



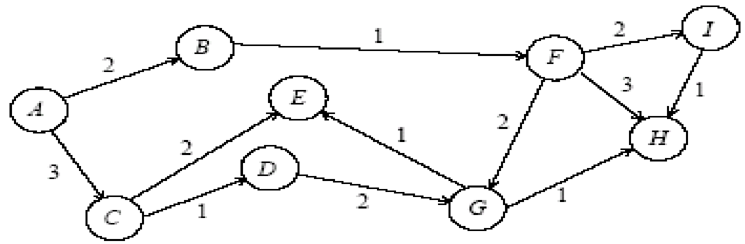
Matrix-Multiplication Algorithm

•Consider the multiplication of the weighted adjacency matrix with itself.

•The product of weighted adjacency matrix with itself returns a matrix that contains shortest paths of length 2 between any pair of nodes.

•It follows that A^n contains all shortest paths.

Matrix-Multiplication Based Algorithm



$$A^1 = \begin{pmatrix} 0 & 2 & 3 & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & 1 & \infty & \infty & \infty \\ \infty & \infty & 0 & 1 & 2 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty & 2 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & \infty & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & 5 & 6 & 5 \\ \infty & 0 & \infty & \infty & 4 & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & 4 & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

$$A^8 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & 5 & 6 & 5 \\ \infty & 0 & \infty & \infty & 4 & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & 4 & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

Continue..

•A^n is computed by doubling powers - i.e., as A, A^2, A^4, A^8, and so on.

•We need $\log n$ matrix multiplications, each taking time $O(n^3)$.

•The serial complexity of this procedure is $O(n^3 \log n)$.

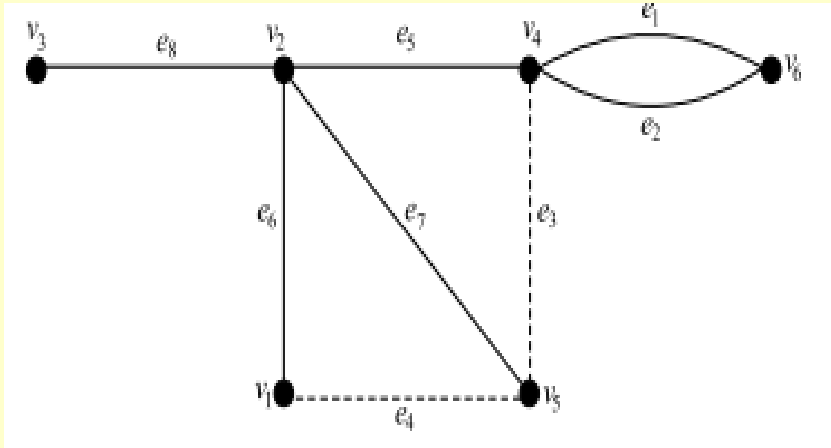
•This algorithm is not optimal, since the best known algorithms have complexity $O(n^3)$.

Path Matrix

Let G be a graph with m edges,u and v vertices.The path matrix P(u, v) = [pij]q×m, where q is the number of different paths between u and v.

pij =1,if jth edge lies in the ith path,

0, otherwise .



The different paths between the vertices v_3 and v_4 are

$$p_1 = \{e_8, e_5\},\ p_2 = \{e_8, e_7, e_3\} \text{ and } p_3 = \{e_8, e_6, e_4, e_3\}.$$

The path matrix for v_3, v_4 is given by

$$P(v_3, v_4) = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

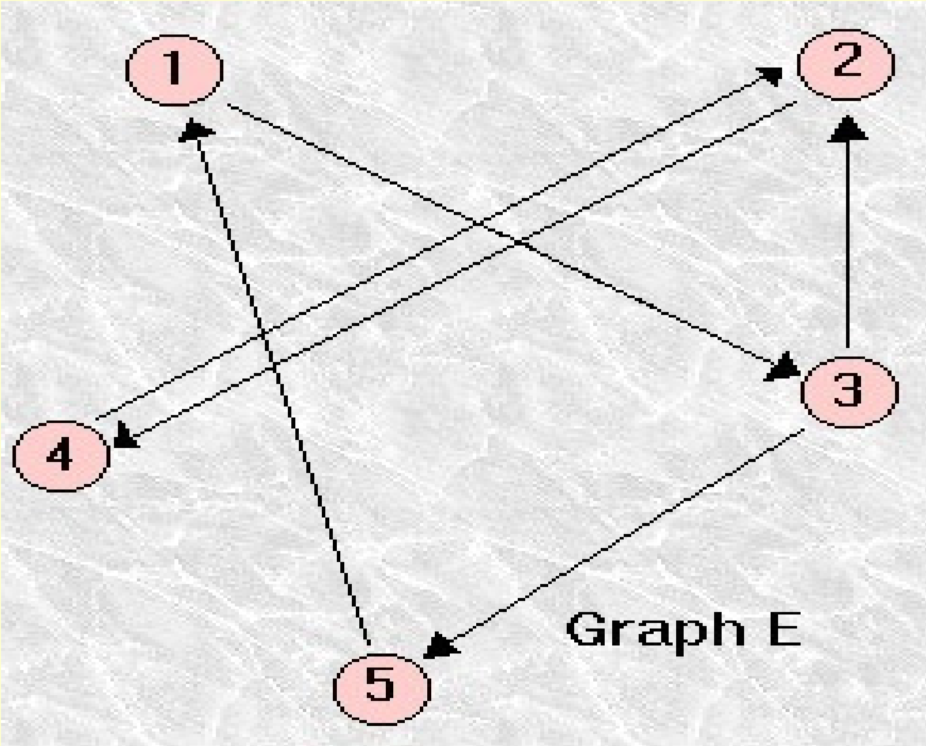
Warshall’s Algorithm

• Warshall's Algorithm is used to compute the existence of paths within a digraph using Boolean operators and matrices.

• It is used for finding shortest paths in a weighted graph with positive or negative edge weights (but with no negative cycles) and also for finding transitive closure of a relation R .

•Complexity of the algorithm is O(|N^3|),where N is number of nodes of the graph.

Warshall’s Algorithm



A	1	2	3	4	5
1	0	0	1	0	0
2	0	0	0	1	0
3	0	1	0	0	1
4	0	1	0	0	0
5	1	0	0	0	0

Begin by creating an adjacency matrix **A** for Graph **E** - instead of using weights,we will use Boolean operators.If there is a path, enter a 1 in matrix **A**, and enter 0 if no path exists.

Continue..

•This matrix tells us whether or not there is a path ***p*** of length 1 between two adjacent nodes.

•Building upon matrix **A**, we will create a new matrix $\mathbf{A^1}$, for which we will choose 1 vertex to act as a *pivot* - an intermediate point between 2 other vertices.

•Initially, we will chose vertex 1 as pivot for $\mathbf{A^1}$.

•For vertices $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$,

$\boldsymbol{p}^{(1)}{}_{i\,j}$ is 1, if there exists an edge between vertices $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$, or if there is a path of length ≥ 2 from $\boldsymbol{v}_i$ to $\boldsymbol{v}_1$ and from $\boldsymbol{v}_1$ to $\boldsymbol{v}_j$.

else 0, if there is no path.

Matrix A^1

- Begin by scanning column 1 of matrix **A**;only vertex 5 connects $\mathbf{v}_i$ to $\mathbf{v}_1$.

- Now scan row 1,the only path from $\mathbf{v}_1$ to $\mathbf{v}_j$ is to vertex 3.

- So, path of length 2 lies between $\mathbf{v}_5$ and $\mathbf{v}_3$, we update matrix $\mathbf{A}^1$ accordingly.

A^1	1	2	3	4	5
1	0	0	1	0	0
2	0	0	0	1	0
3	0	1	0	0	1
4	0	1	0	0	0
5	1	0	1	0	0

Matrix A^2

- Next create matrix $\mathbf{A}^2$, using vertex 2 as the pivot point.

- Begin by scanning *column 2* of matrix **A**; the $\boldsymbol{v}_i$ which connect to $\boldsymbol{v}_2$ are vertices 3 and 4.

- Now scan *row 2*; only 1 path from $\boldsymbol{v}_2$ exists to $\boldsymbol{v}_j$ = vertex 4.

- Newly added paths have been highlighted in gray.

$\mathbf{A}^2$	1	2	3	4	5
1	0	0	1	0	0
2	0	0	0	1	0
3	0	1	0	1	1
4	0	1	0	1	0
5	1	0	1	0	0

Matrix A^3

•Matrix $\mathbf{A}^3$ use vertex 3 as the pivot point.

•Vertices 1 and 5 have a path to 3.

•Now, scanning row 3, v_3 connects to vertices 2, 4, 5. Paths established :-

v_1 to v_2
v_1 to v_4
v_1 to v_5
v_5 to v_2
v_5 to v_4
v_5 to v_5

$\mathbf{A}^3$	1	2	3	4	5
1	0	1	1	1	1
2	0	0	0	1	0
3	0	1	0	1	1
4	0	1	0	1	0
5	1	1	1	1	1

Matrix A^4

Some paths have exceeded length 2 because the newly established paths are not using just 3 as a pivot point, but also the previous pivots points.

•Now,we will be creating 2 more adjacency matrices, $\mathbf{A}^4$ and $\mathbf{A}^5$.

•For $\mathbf{A}^4$, first scan column 4.

•All vertices now have a path to vertex 4.

•Scanning row 4, we see that 4 has a path only to vertex 2, indicating that all vertices have a path to 2.

•The only vertex which doesn't already have a path to vertex 2 is 2 itself.

A^4	1	2	3	4	5
1	0	1	1	1	1
2	0	1	0	1	0
3	0	1	0	1	1
4	0	1	0	1	0
5	1	1	1	1	1

If a graph has ***n*** vertices, it will require ***n*** matrices to produce $\mathbf{A}^n = \mathbf{P}^n$, where $\mathbf{P}^n$ is the path matrix.

Matrix A^5

•Now, scan column 5 to see that vertices 1, 3 and 5 all have paths to vertex 5.

• Scanning row 5 indicates that 5 has a path to all other vertices.

•Consequently, we add 1's to rows 1, 3 and 5 to reflect that vertices 1, 3 and 5 have paths to all other vertices.

A^5	1	2	3	4	5
1	1	1	1	1	1
2	0	1	0	1	0
3	1	1	1	1	1
4	0	1	0	1	0
5	1	1	1	1	1

This completes the path matrix for Graph E.

Warshall's Algorithm for computing a path matrix

procedure *Warshall*
(A: BoolMatrix; /*Input, the adjacency matrix of a given graph*/
var P: BoolMatrix; /*Output, the path matrix of the graph*/
n: **integer**); /*Input, the size of the matrix (i.e., the number of vertices)*/
int i, j, k;
Begin
for i :=1 **to** n **do**
    for j := 1 **to** n **do**
        P[i, j] := A[i, j]; /*Step 1: Copy adjacency into path matrix*/
for k := 1 **to** n **do** /*Step 2: Allow vertex k as a pivot point*/
for i := 1 **to** n **do** /*Step 3: Process rows*/
  for j := 1 **to** n **do** /*Step 4: Process columns*/
 P[i, j] := P[i, j] **or** (P[i, k] **and** P[k, j]) /*Step 5*/
end;